AMENABILITY AND THE LIOUVILLE PROPERTY

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Dedicated to Hillel Furstenberg

Abstract. We present a new approach to the amenability of groupoids (both in the measure theoretical and the topological setups) based on using Markov operators. We introduce the notion of an invariant Markov operator on a groupoid and show that the Liouville property (absence of non-trivial bounded harmonic functions) for such an operator implies amenability of the groupoid. Moreover, the groupoid action on the Poisson boundary of any invariant operator is always amenable. This approach subsumes as particular cases numerous earlier results on amenability for groups, actions, equivalence relations and foliations. For instance, we establish in a unified way topological amenability of the boundary action for isometry groups of Gromov hyperbolic spaces, Riemannian symmetric spaces and affine buildings.

Contents

Introduction

The notion of *amenability* for groups is, from the analytical point of view, the most natural generalization of finiteness or compactness. *Amenable groups* are those which admit an *invariant mean* (rather than an invariant probability measure, which is the case for compact groups). There are many other equivalent definitions of an amenable group, see [Gre69], [Pat88], [Pie84]. Among the most constructive is the one formulated in terms of existence of *approximatively invariant sequences of probability measures* on the group (*Reiter's condition*), whereas one of the main applications of amenability is the *fixed point property* for affine actions of amenable groups on compact spaces.

It turns out that non-amenable groups may still have actions which look like actions of amenable groups. This observation led Zimmer [Zim77], [Zim78] to introduce the notion of an *amenable action*. In the same way as with groups, there are several definitions of an amenable action. In particular, amenable actions can be characterized both in terms of a fixed point property (this was the original definition of Zimmer) and in terms of existence of a sequence of approximatively equivariant maps from the action space to the

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space of probability measures on the group (this is an analogue of Reiter's condition). Yet another generalization is the notion of amenability for *equivalence relations* and *foliations* [CFW81]. All these objects can be considered as *measured groupoids*, and the notion of amenability in each particular case is a specialization of the general notion of an *amenable measured groupoid*. There is also a similar notion of an *amenable topological groupoid* (defined in the topological rather than measure theoretical category) as well. The general references for the theory of amenable groupoids are [Ren80], [ADR00], [CHLI02].

A *groupoid* $\mathbf{G}$ is a small category in which each morphism is an isomorphism, so that it is determined by its *set of morphisms* (also denoted $\mathbf{G}$) fibered over the *set of objects* $\mathbf{G}^{(0)}$ via the source s and the target t maps. If $\mathbf{G}$ and $\mathbf{G}^{(0)}$ are topological spaces and the structure maps are continuous, then $\mathbf{G}$ is called a *topological groupoid*. Similarly, if $\mathbf{G}, \mathbf{G}^{(0)}$ are Borel spaces and the structure maps are Borel, then $\mathbf{G}$ is a *Borel groupoid*. The basic examples of groupoids are those associated with groups (of course!), group actions and equivalence relations.

The fibers $\mathbf{G}_x, \mathbf{G}^x$ of the source and the target maps, respectively, are moved around by morphisms; for instance, $\mathbf{g}\mathbf{G}^{\mathsf{s}(\mathbf{g})} = \mathbf{G}^{\mathsf{t}(\mathbf{g})}$ for any $\mathbf{g} \in \mathbf{G}$. A *Haar system* (defined by analogy with Haar measures on groups) on a Borel groupoid $\mathbf{G}$ is a Borel system $\lambda = \{\lambda^x\}$ of measures on the fibers of the target map which is *invariant* in the sense that $\mathbf{g}\lambda^{\mathsf{s}(\mathbf{g})} = \lambda^{\mathsf{t}(\mathbf{g})}\ \forall\, \mathbf{g} \in \mathbf{G}$. In order to define a *measured groupoid* in addition to a Haar system λ one has to specify a Borel measure μ on the space of objects $\mathbf{G}^{(0)}$ which is *quasi-invariant* with respect to the system λ. The latter means that the *global measure* $\lambda \star \mu$ on $\mathbf{G}$ obtained by integrating the fiberwise measures λ^x against the measure μ on the base is quasi-invariant with respect to the flip $\mathbf{g} \mapsto \mathbf{g}^{-1}$.

Groupoids with a finite Haar system are similar to compact groups [Hah78]. In the same way as in the group case, the immediate generalization of the existence of a finite Haar system is existence of an *approximatively invariant* sequence $\theta_n = \{\theta_n^x\}$ of probability measures on the fibers $\mathbf{G}^x$, i.e., such that $\left\|\mathbf{g}\theta_n^{\mathsf{s}(\mathbf{g})} - \theta_n^{\mathsf{t}(\mathbf{g})}\right\| \to 0$. Groupoids with this property are called *amenable*. Of course, some further assumptions have to be made and the notion of convergence itself has to be specified depending on whether we deal with measured or topological groupoids. For measured groupoids one requires the systems θ_n to be Borel and absolutely continuous with respect to the Haar system λ, and the convergence to be weak* in the space $L^\infty(\mathbf{G}, \lambda \star \mu)$, whereas for topological groupoids one requires the systems θ_n to be continuous, and the convergence to be uniform on compact subsets of $\mathbf{G}$.

The definition of an amenable measured groupoid in terms of approximatively invariant sequences was given by Renault shortly after the work of Zimmer [Ren80, Lemma II.3.4]. However, until recently it remained relatively unknown to specialists working on amenability of equivalence relations and group actions in the measure theoretical setup, where either the original definition of Zimmer or the definition in terms of existence of a $\mathbf{G}$-invariant mean $L^\infty(\mathbf{G}) \to L^\infty\big(\mathbf{G}^{(0)}\big)$ were used, see [CFW81], [CW89], [AEG94]; in the context of equivalence relations it was reintroduced in [Kai97] (at the time I was not aware of Renault's work). Being very constructive, Renault's definition significantly simplifies and clarifies proofs of amenability. Compare, for instance, the original proof of amenability of the boundary action for isometry groups of Gromov hyperbolic spaces due to Adams [Ada94], [Ada96] with the recent much shorter arguments in [Ger00] and [Kai03].

Another advantage of the definition of Renault is that it can easily be adapted to the topological setup, where it found numerous applications to the theory of C^*-algebras, see [AD87], [Hig00], [HR00], [AD02], [Val02], [CEOO03].

The classical *Liouville theorem* asserts absence of bounded harmonic functions on the Euclidean space. The notion of a *harmonic function* (one which satisfies the mean value property with respect to the transition probabilities) naturally carries over to general Markov operators acting on a measure space. Such an operator is said to have the *Liouville property* if it has no non-constant bounded harmonic functions. The link between the Liouville property and amenability is based on the so-called *0–2 laws* for Markov operators due to Derriennic [Der76] (also see [Kai92]), which, in particular, assert that the Liouville property is equivalent to asymptotic independence of n-step transition probabilities of the initial distribution. This is precisely what is needed for proving amenability by constructing approximatively invariant sequences of probability measures. Yet another, less constructive, way of connecting the Liouville property with amenability consists in using *measure-linear means* on $\mathbb{Z}_+$. Any such mean applied along the sample paths of a Liouville Markov chain provides a projection from the space of functions on the state space onto constants which is invariant with respect to all the symmetries of the operator, see [CFW81], [LS84], [KF98].

In order to realize this idea we introduce the notion of an *invariant Markov operator* on a groupoid $\mathbf{G}$ by analogy with invariant Markov operators on groups (corresponding to the usual random walks on groups). The transition probabilities of such an operator satisfy the equivariance condition $\pi_{\mathbf{g}'\mathbf{g}} = \mathbf{g}'\pi_{\mathbf{g}}$ for any $\mathbf{g}', \mathbf{g} \in \mathbf{G}$ whenever the composition $\mathbf{g}'\mathbf{g}$ is well-defined. Invariant Markov operators on $\mathbf{G}$ are in one-to-one correspondence with systems of probability measures on the fibers of the target map $\mathsf{t} : \mathbf{G} \to \mathbf{G}^{(0)}$, and the product of two invariant Markov operators corresponds to the usual convolution operation for such systems of measures (or, for their densities with respect to a Haar system in the absolutely continuous case).

The definition of an invariant Markov operator implies, in particular, that any transition probability $\pi_{\mathbf{g}}$ is concentrated on the corresponding fiber $\mathbf{G}^{\mathsf{t}(\mathbf{g})}$, so that the sample paths of the associated Markov chain are confined to the fibers of the target map. Therefore, an invariant Markov operator on $\mathbf{G}$ can be considered as a $\mathbf{G}$-invariant collection of Markov operators on the fibers of the target map.

As particular cases this notion includes *covering Markov operators* (for instance, the ones associated with the *Brownian motion on covering manifolds*) and, more generally, families of their *conditional operators* with respect to the Poisson boundary (or its equivariant quotients), the Markov operators associated with *random walks on equivalence relations* and *leafwise Brownian motion on foliations* as well as all known models of *randomization* of the usual random walk on a group (random walks in *random environment*, with *internal degrees of freedom*, with *random transition probabilities*).

We require invariant Markov operators on groupoids to act on the fiberwise L^∞ spaces with respect to a Haar system, and call such an operator *fiberwise Liouville* if its restrictions to the fibers of the target map have the Liouville property. The main result of the paper is that *if a groupoid carries a fiberwise Liouville invariant Markov operator then it is amenable.* We prove it both for measured and topological groupoids (Theorem 4.2 and Theorem 6.1, respectively). Particular cases of this result were earlier established for groups [Aze70], [Fur73], for equivalence relations and foliations [CFW81], for the Brownian motion on covering manifolds [LS84], for general covering Markov operators [Kai95] as well as for various models of randomization of the usual random walk on a discrete group, see [Sun87], [KKR02].

It is plausible that the converse is also true, at least for measured groupoids, namely, that *any amenable groupoid carries a fiberwise Liouville invariant Markov operator*. This

is known to be the case for groups [Ros81], [KV83] (it had been previously conjectured by Furstenberg [Fur73]), for discrete equivalence relations (in view of the Connes–Feldman–Weiss theorem on the coincidence of amenability and hyperfiniteness [CFW81]) and for group actions (in a somewhat weaker form, though; by [EG93], [AEG94] any amenable measure class preserving action of a locally compact group G can be realized as the action on the Poisson boundary of an appropriate G-invariant operator on the product of G by a countable set).

As a specialization of our main result to the case of groupoids associated with group actions we show (once again both in the measure theoretical and topological setups, Theorem 4.8 and Theorem 6.3, respectively) that *if there exists an equivariant map assigning to the points from the action space X minimal harmonic functions of a certain G-invariant Markov operator on another space S (endowed with a proper action of the group G), then the action of G on X is amenable.* The reason is that the *Doob transforms* of the original operator determined by these minimal harmonic functions have the Liouville property. In the particular case of the random walk on a countable group with a finitely supported transition probability this result was also independently obtained in [BG02].

Typically, such a situation arises when X is a certain *boundary* of the space S. Identification of the space of minimal harmonic functions of a G-invariant Markov operator (in other words, of the *Poisson boundary* in the measure theoretical setup or the *minimal Martin boundary* in the topological setup) is, in general, a difficult task (e.g., see [Kai96]). However, natural geometrical boundaries are known to produce minimal harmonic functions in several situations of hyperbolic flavour: for *simply connected negatively curved manifolds with pinched sectional curvatures* and, more generally, *Gromov hyperbolic spaces*; for *Riemannian symmetric spaces*; for *affine buildings*. Therefore, our result immediately implies the strongest possible *topological amenability of the boundary actions of the groups of isometries* of these spaces (Theorem 6.6). In particular, these groups are *amenable at infinity*. This provides a unified generalization of numerous earlier results on amenability of boundary actions [Bow77], [Ver78], [Zim84], [Spa87], [SZ91], [Ada94], [Ada96], [RS96], [RR96], [CR03].

Yet another particular case is the *measure theoretical amenability of the action of a locally compact group G on the Poisson boundary* either of a usual random walk on G or of a certain G-invariant chain on a G-space, which was earlier established in [Zim78] and [CW89] (also see [EG93], [AEG94]).

More generally, following the considerations for covering Markov operators in [Kai95] we introduce the notion of the *Poisson extension* of an invariant Markov operator P on a groupoid $\mathbf{G}$. This is the measured groupoid associated with the action of $\mathbf{G}$ on the Poisson boundary of the operator P. We prove that *the Poisson extension is amenable* (Theorem 5.2), which subsumes Theorem 4.2 and provides a generalization of the above results on the amenability of the Poisson boundary in the group case.

These results were presented at a number of seminars and conferences (University of Chicago 1997, ENS Lyon 1998, University of Genova 1999, Rokhlin memorial conference, St. Petersburg 1999, University of Orleans 2000, University of Neuchâtel 2001, Caltech 2002). In particular, at the seminar of Anantharaman–Delaroche and Renault in Orleans in May 2000 a draft version of the present article was circulated.

The paper has the following structure. In Section 1 and Section 2 we recall main definitions concerning groupoids and their amenability. The exposition here is mostly based on the books [Ren80] and [ADR00]. In Section 3 we introduce the notion of an invariant Markov operator on a groupoid and discuss various examples of such operators.

In Section 4 we prove amenability of measured groupoids with fiberwise Liouville invariant Markov operators, and in Section 5 we prove amenability of the Poisson extension of an invariant Markov operator on a measured groupoid. Finally, in Section 6 we establish analogues of these results in the topological category.

The spirit of this paper owes a lot to the work and ideas of *Hillel Furstenberg*. It was him who laid the foundation of the modern probabilistic boundary theory of algebraic structures, and, in particular, established the first results relating amenability to the Liouville property for random walks on groups — which are the starting point of the present article. I dedicate this paper to him with admiration.

1. Groupoids

1.A. General definitions. Let us first recall the definition of a *groupoid* as a *small category in which each morphism is an isomorphism.*

In other words, a groupoid $\mathbf{G}$ is determined by a *set of objects* (also called the *set of units*)

$$\mathsf{Obj}\,\mathbf{G} = \mathbf{G}^{(0)}$$

and a *set of morphisms*

$$\mathsf{Mor}\,\mathbf{G} \cong \mathbf{G}$$

(which, following the well-established tradition, we shall usually denote just by $\mathbf{G}$) endowed with the *source* and *target* maps

$$\mathsf{s}, \mathsf{t} : \mathbf{G} \to \mathbf{G}^{(0)} .$$

Denote by

$$\mathbf{G}^{(2)} = \{(g_1, g_2) \in \mathbf{G} \times \mathbf{G} : \mathsf{s}(\mathbf{g}_1) = \mathsf{t}(\mathbf{g}_2)\}$$

the set of *composable pairs* in $\mathbf{G}$. The composition is a map

$$\mathbf{G}^{(2)} \to \mathbf{G} , \qquad (\mathbf{g}_1, \mathbf{g}_2) \mapsto \mathbf{g}_1\mathbf{g}_2$$

such that

$$\mathsf{s}(\mathbf{g}_1\mathbf{g}_2) = \mathsf{s}(\mathbf{g}_2) , \qquad \mathsf{t}(\mathbf{g}_1\mathbf{g}_2) = \mathsf{t}(\mathbf{g}_1) .$$

Remark 1.1. Our notations match those used for left actions of groups: $(g_1g_2)x = g_1(g_2x)$, i.e., g_2 is applied “first”. Alternatively, one could choose the “postfix” notation (which also has some euristic advantages) corresponding to the right actions with $x(g_1g_2) = (xg_1)g_2$.

The composition has the usual properties. Namely, there is an embedding

$$\boldsymbol{\varepsilon} : \mathbf{G}^{(0)} \to \mathbf{G} ,$$

which associates to any object $x \in \mathbf{G}^{(0)}$ the *identical automorphism* $\boldsymbol{\varepsilon}_x$ such that

$$\mathsf{s}(\boldsymbol{\varepsilon}_x) = \mathsf{t}(\boldsymbol{\varepsilon}_x) = x \qquad \forall\, x \in \mathbf{G}^{(0)}$$

(usually we shall just identify x and $\boldsymbol{\varepsilon}_x$). For any $\mathbf{g} \in \mathbf{G}$ there is a unique *inverse morphism* $\mathbf{g}^{-1}$ with the property that

$$\mathsf{s}(\mathbf{g}^{-1}) = \mathsf{t}(\mathbf{g}) , \qquad \mathsf{t}(\mathbf{g}^{-1}) = \mathsf{s}(\mathbf{g})$$

and

$$\mathbf{g}\mathbf{g}^{-1} = \boldsymbol{\varepsilon}_{\mathsf{t}(\mathbf{g})} , \qquad \mathbf{g}^{-1}\mathbf{g} = \boldsymbol{\varepsilon}_{\mathsf{s}(\mathbf{g})} .$$

Finally, the composition (when well-defined) is associative.

The fibers of the source and target maps are denoted

$$\mathbf{G}_x = \mathsf{s}^{-1}(x) , \qquad \mathbf{G}^x = \mathsf{t}^{-1}(x) ,$$

respectively. More generally, for any two subsets $X, Y \subset \mathbf{G}^{(0)}$ and a subset $A \subset \mathbf{G}$, we put

$$A_X = A \cap \mathsf{s}^{-1}(X) , \qquad A^Y = A \cap \mathsf{t}^{-1}(Y) , \qquad A_X^Y = A_X \cap A^Y .$$

The *isotropy group* of an object $x \in \mathbf{G}^{(0)}$ is then $\mathbf{G}_x^x$ (whose unit is $\boldsymbol{\varepsilon}_x$; this is the reason why $\mathbf{G}^{(0)}$ is called the set of units). The set

$$\mathbf{G}' = \{\mathbf{g} \in \mathbf{G} : \mathsf{s}(\mathbf{g}) = \mathsf{t}(\mathbf{g})\} = \bigcup_{x \in \mathbf{G}^{(0)}} \mathbf{G}_x^x$$

is called the *isotropy bundle*. The groupoid $\mathbf{G}$ determines the associated *orbit equivalence relation*

$$R_{\mathbf{G}} = \{(\mathsf{s}(\mathbf{g}), \mathsf{t}(\mathbf{g})) : \mathbf{g} \in \mathbf{G}\} \subset \mathbf{G}^{(0)} \times \mathbf{G}^{(0)}$$

on the space of objects $\mathbf{G}^{(0)}$. Its classes are called *orbits* of $\mathbf{G}$ in $\mathbf{G}^{(0)}$. Therefore, any groupoid can be considered as an extension of its orbit equivalence relation by the isotropy bundle.

1.B. Examples of groupoids. The most basic examples are:

(i) *Groups.* Any group G can be in a trivial way considered as a groupoid by putting

$$\mathbf{G} = G \quad \text{and} \quad \mathbf{G}^{(0)} = \{o\}$$

for a single point o with $\mathsf{s}(\mathbf{g}), \mathsf{t}(\mathbf{g}) \equiv o$ and the same composition rule in $\mathbf{G}$ as in the original group G. Then clearly $\mathbf{G}$ consists just of the isotropy group $\mathbf{G}_o^o$ isomorphic to G.

(ii) *Equivalence relations.* If $R \subset X \times X$ is an equivalence relation on a set X, then for the associated groupoid

$$\mathbf{G} = R \quad \text{and} \quad \mathbf{G}^{(0)} = X$$

with source and target maps

$$\mathsf{s}(x, y) = y , \quad \mathsf{t}(x, y) = x ,$$

respectively, the composition

$$(x, y)(y, z) = (x, z) ,$$

and the inverse map

$$(x, y)^{-1} = (y, x) .$$

The embedding $\boldsymbol{\varepsilon} : \mathbf{G}^{(0)} \to \mathbf{G}$ is diagonal, i.e.,

$$\boldsymbol{\varepsilon}_x = (x, x) .$$

All isotropy groups are trivial, whereas the orbit equivalence relation $R_{\mathbf{G}}$ on the set of objects $\mathbf{G}^{(0)} \cong X$ is precisely the original equivalence relation R.

(iii) *Group actions.* Let a group G acts (on the left) on a set X. For the associated groupoid

$$\mathbf{G} = \{(gx, g, x) : g \in G, x \in X\} \quad \text{and} \quad \mathbf{G}^{(0)} = X$$

with the source and target maps

$$\mathsf{s}(gx, g, x) = x , \quad \mathsf{t}(gx, g, x) = gx ,$$

respectively, the embedding

$$\boldsymbol{\varepsilon}_x = (x, e, x)$$

(here e is the identity of the group G), the inverse map

$$(gx, g, x)^{-1} = (x, g^{-1}, gx) ,$$

and the composition

$$(g_1 g_2 x, g_1, g_2 x)(g_2 x, g_2, x) = (g_1 g_2 x, g_1 g_2, x) \ .$$

The isotropy group $\mathbf{G}_x^x$ is then isomorphic to the stabilizer

$$\mathrm{Stab}_x = \{g \in G : gx = x\}$$

of the point x in the group G, and the orbit equivalence relation $R_{\mathbf{G}}$ is the usual orbit equivalence relation of the action of the group G. In particular, for the trivial action of the group G on the singleton $\{o\}$ we obtain the groupoid (i) associated with the group G. If the action is free, then we obtain the groupoid (ii) associated with the orbit equivalence relation of the action of the group G on X.

See [Ren80], [ADR00] and [CHLI02] for more examples of groupoids, for instance, those arising from transformation pseudogroups and foliated manifolds. Some further examples (together with a description of invariant Markov operators on them) are also given in Section **3.D**.

1.C. **Homogeneous spaces.** By analogy with the group case one can also define the notion of a $\mathbf{G}$-space for a groupoid $\mathbf{G}$. A (left) $\mathbf{G}$-*space* consists of a set X endowed with a *projection map* $\mathsf{t} = \mathsf{t}_X : X \to \mathbf{G}^{(0)}$ and an *action map*

$$\{(\mathbf{g}, x) \in \mathbf{G} \times X : \mathsf{s}(\mathbf{g}) = \mathsf{t}(x)\} \to X \ , \qquad (\mathbf{g}, x) \mapsto \mathbf{g}x$$

with natural properties, more precisely,

$$\mathsf{t}(\mathbf{g}x) = \mathsf{t}(\mathbf{g}) \ , \quad \varepsilon_{\mathsf{t}(x)} x = x \ , \quad \text{and} \quad \mathbf{g}_1(\mathbf{g}_2 x) = (\mathbf{g}_1 \mathbf{g}_2) x$$

whenever the corresponding products are well-defined.

Any $\mathbf{G}$-space gives rise (in the same way as for usual group actions, see example (iii) in Section **1.B** above) to the associated groupoid called the *semi-direct product* of $\mathbf{G}$ and X and denoted $\mathbf{G} \ltimes X$. Namely,

$$\mathsf{Mor}(\mathbf{G} \ltimes X) = \{(\mathbf{g}x, \mathbf{g}, x) : \mathsf{s}(\mathbf{g}) = \mathsf{t}(x)\} \quad \text{and} \quad \mathsf{Obj}(\mathbf{G} \ltimes X) = X \ ,$$

whereas the structure maps are defined in precisely the same way as in example (iii) from Section **1.B**. In particular, for any groupoid $\mathbf{G}$ its set of objects $\mathbf{G}^{(0)}$ can be in a natural way considered as a $\mathbf{G}$-space, and the semi-direct product $\mathbf{G} \ltimes \mathbf{G}^{(0)}$ is isomorphic to $\mathbf{G}$.

1.D. **Measured groupoids.** Any morphism $\mathbf{g} \in \mathbf{G}$ determines the bijection

$$\mathbf{G}^{\mathsf{s}(\mathbf{g})} \to \mathbf{G}^{\mathsf{t}(\mathbf{g})} \ , \qquad \mathbf{h} \mapsto \mathbf{gh} \ .$$

This observation prompts one to define (by the analogy with the classical notion of a Haar measure on a locally compact group) the notion of a *Haar system of measures* on a groupoid $\mathbf{G}$ as a family of measures $\lambda = \{\lambda^x\}$ on the fibers $\mathbf{G}^x \subset \mathbf{G}$ of the target map $\mathsf{t} : \mathbf{G} \to \mathbf{G}^{(0)}$ which is $\mathbf{G}$-*invariant* in the sense that

$$\mathbf{g}\lambda^{\mathsf{s}(\mathbf{g})} = \lambda^{\mathsf{t}(\mathbf{g})} \qquad \forall \mathbf{g} \in \mathbf{G} \ . \tag{1.2}$$

More precisely, we shall say that a groupoid $\mathbf{G}$ is *Borel* if it is endowed with a Borel structure such that the structure maps are Borel, where $\mathbf{G}^{(0)}$ and $\mathbf{G}^{(2)}$ are given the Borel structures induced by $\mathbf{G}$ and $\mathbf{G} \times \mathbf{G}$, respectively. Then a family of measures $\{\lambda^x\}$ on $\mathbf{G}$ indexed by points $x \in \mathbf{G}^{(0)}$ is called *Borel* if for any non-negative Borel function f on $\mathbf{G}$ the function on $\mathbf{G}^{(0)}$

$$\lambda(f) : x \mapsto \langle \lambda^x, f \rangle \tag{1.3}$$

is also Borel. Such a family is called *proper* if there exists a Borel function f with $\lambda(f) \equiv 1$. Then a *Borel Haar system* on a Borel groupoid $\mathbf{G}$ is a $\mathbf{G}$-invariant proper Borel family of measures $\{\lambda^x\}$ concentrated on the sets $\mathbf{G}^x$ (i.e., $\lambda^x(\mathbf{G} \setminus \mathbf{G}^x) = 0$ for any $x \in \mathbf{G}^{(0)}$).

A measure ν on $\mathbf{G}$ is called *quasi-symmetric* if it is quasi-invariant with respect to the inverse map $\mathbf{g} \mapsto \mathbf{g}^{-1}$. One can integrate a Borel system of measures $\lambda = \{\lambda^x\}$ on the fibers $\mathbf{G}^x$ of the target map $\mathsf{t} : \mathbf{G} \to \mathbf{G}^{(0)}$ with respect to any Borel measure μ on $\mathbf{G}^{(0)}$, which gives the Borel measure $\lambda \star \mu$ on $\mathbf{G}$. Then the measure μ is called *quasi-invariant* with respect to the system λ if the measure $\lambda \star \mu$ is quasi-symmetric.

Finally, a *measured groupoid* is a triple $(\mathbf{G}, \lambda, \mu)$, where $\mathbf{G}$ is a Borel groupoid, λ is a Borel Haar system, and μ is a measure on $\mathbf{G}^{(0)}$ quasi-invariant with respect to λ. Actually, in this definition we only need the class of the measure μ, and below we shall also apply the term "measured groupoid" to the situation when just a quasi-invariant measure class (rather than a specific measure) on the space of objects is given.

Given a Borel groupoid $\mathbf{G}$ endowed with a Borel Haar system λ and a Borel $\mathbf{G}$-space X, the projection map $X \mapsto \mathbf{G}^{(0)}$ allows one to lift the system λ to a Borel Haar system (also denoted λ) of the semi-direct product $\mathbf{G} \ltimes X$. A Borel measure μ on X is then called *quasi-invariant* with respect to $(\mathbf{G}, \lambda)$ if the measure $\lambda \star \mu$ on the groupoid $\mathbf{G} \ltimes X$ is quasi-symmetric, i.e., if the triple $(\mathbf{G} \ltimes X, \lambda, \mu)$ is a measured groupoid [ADR00, Definition 3.1.1] (in the case when $X = \mathbf{G}^{(0)}$ this definition clearly agrees with the previous definition of a quasi-invariant measure on the space of objects, see Section **1.C**).

In particular, any measured groupoid $(\mathbf{G}, \lambda, \mu)$ naturally acts on itself, and the measure $\lambda \star \mu$ on $\mathbf{G}$ is quasi-invariant with respect to $(\mathbf{G}, \lambda)$ [Hah78, Proposition 3.4] (the associated semi-direct product $\mathbf{G} \ltimes \mathbf{G}$ is isomorphic to $\mathbf{G}^{(2)}$).

1.E. Topological groupoids. In the *topological setting*, one assumes that the groupoid $\mathbf{G}$ is a topological space and that the structure maps are continuous, where $\mathbf{G}^{(2)}$ has the topology induced by $\mathbf{G} \times \mathbf{G}$, and $\mathbf{G}^{(0)}$ has the topology induced by $\mathbf{G}$. Furthermore, the source and the target maps are surjective and open. Similar assumptions are made in the definition of a continuous $\mathbf{G}$-space. We shall be concerned exclusively with topological groupoids and continuous $\mathbf{G}$-spaces which are *second countable, locally compact and Hausdorff*.

We shall use the standard definition of a *continuous Haar system* λ for a locally compact groupoid $\mathbf{G}$: continuity in this situation means that for any compactly supported continuous function f on $\mathbf{G}$ the associated function $\lambda(f)$ (1.3) on $\mathbf{G}^{(0)}$ is also continuous.

2. Amenability

2.A. Amenable groups. The notion of *amenability* was first introduced for groups and is the most natural, from the analytical point of view, generalization of finiteness or compactness. Namely, compact groups are distinguished within the class of all locally compact topological groups by the property that they carry a *finite* invariant measure.

Recall that a *mean* on a measure space (X, m) is a positive normalized (hence, continuous) linear functional on the space $L^\infty(X, m)$, or, equivalently, a *finitely additive* probability measure on X absolutely continuous with respect to m. In a similar way one defines means with values in $L^\infty(X', m')$ for any quotient space (X', m') of (X, m).

For an *amenable group* G one requires existence of an *invariant mean* on $L^\infty(G)$ instead of a probability measure (this is the classical definition of amenability), or, equivalently, by *Reiter's condition*, existence of a sequence of probability measures θ_n on G which is

approximatively invariant in the sense that

$$\|g\theta_n - \theta_n\| \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall\, g \in G \; .$$

Reiter's condition is one of the most constructive among a host of other (equivalent) definitions of amenability, see [Gre69], [Pie84], [Pat88].

2.B. **Amenable groupoids.** The notion of amenability can be naturally generalized to groupoids (including groups as a particular case). A measured groupoid $(\mathbf{G}, \lambda, \mu)$ is called *amenable* if there exists a $\mathbf{G}$-invariant mean

$$\mathbf{\Pi} : L^\infty(\mathbf{G}, \lambda \star \mu) \to L^\infty(\mathbf{G}^{(0)}, \mu) \; ,$$

where $\mathbf{G}^{(0)}$ is considered as the quotient of $\mathbf{G}$ under the target map [ADR00, Definition 3.2.8]. By $\mathbf{G}$-*invariance* we mean that $\mathbf{\Pi}$ commutes with the convolution with any system of finite measures $\theta = \{\theta^x\}$ on the fibers of the target map absolutely continuous with respect to the Haar system, i.e.,

$$\mathbf{\Pi}(\theta * f) = \theta * \mathbf{\Pi}(f) \qquad \forall\, f \in L^\infty(\mathbf{G}, \lambda \star \mu) \; ,$$

where the *convolution* is defined as

$$\theta * f(\mathbf{g}) = \int f(\mathbf{h}^{-1}\mathbf{g})\, d\theta^{\mathsf{t}(\mathbf{g})}(\mathbf{h}) \; .$$

As in the group case, this definition has a constructive counterpart formulated in terms of systems of probability measures $\theta = \{\theta^x\}_{x\in\mathbf{G}^{(0)}}$ on the fibers $\mathbf{G}^x \subset \mathbf{G}$ of the target map $\mathsf{t} : \mathbf{G} \to \mathbf{G}^{(0)}$ which are absolutely continuous with respect to the Haar system $\lambda = \{\lambda^x\}$ and measurable in the sense that the Radon–Nikodym derivative $d\theta^{\mathsf{t}(\mathbf{g})}/d\lambda^{\mathsf{t}(\mathbf{g})}(\mathbf{g})$ is a measurable function on $\mathbf{G}$. Amenability of a measured groupoid $(\mathbf{G}, \lambda, \mu)$ is then equivalent to existence of a sequence θ_n of such systems which is *approximatively invariant* in the sense that

$$\int \left\|\mathbf{g}\theta_n^{\mathsf{s}(\mathbf{g})} - \theta_n^{\mathsf{t}(\mathbf{g})}\right\| f(\mathbf{g})\, d\lambda \star \mu(\mathbf{g}) \underset{n\to\infty}{\longrightarrow} 0 \tag{2.1}$$

for any test function $f \in L^1(\mathbf{G}, \lambda \star \mu)$ (this is a reformulation of condition (iv) from [ADR00, Proposition 3.2.14]).

2.C. **Topological amenability.** In the topological setup, for a locally compact groupoid $\mathbf{G}$, a system of probability measures $\theta = \{\theta^x\}$ on the fibers $\mathbf{G}^x \subset \mathbf{G}$, $x \in \mathbf{G}^{(0)}$ of the target map $\mathsf{t} : \mathbf{G} \to \mathbf{G}^{(0)}$ is called *continuous* if for any continuous function with compact support f the associated function $\theta(f)$ (1.3) is continuous on $\mathbf{G}^{(0)}$. A locally compact groupoid $\mathbf{G}$ is called *topologically amenable* if it admits a sequence θ_n of continuous systems of probability measures on the fibers $\mathbf{G}^x$ which is *topologically approximatively invariant* in the sense that

$$\left\|\mathbf{g}\theta_n^{\mathsf{s}(\mathbf{g})} - \theta_n^{\mathsf{t}(\mathbf{g})}\right\| \underset{n\to\infty}{\longrightarrow} 0$$

uniformly on compacts in $\mathbf{G}$ [ADR00, Definitions 2.2.1 and 2.2.8]. In the presence of a continuous Haar system λ on $\mathbf{G}$ the systems θ_n from the above definition can be chosen in such a way that the Radon–Nikodym derivatives $d\theta_n^{\mathsf{t}(\mathbf{g})}/d\lambda_n^{\mathsf{t}(\mathbf{g})}(\mathbf{g})$ are continuous on $\mathbf{G}$ [ADR00, Proposition 2.2.13].

See [ADR00] for more details on the notion of amenability for measured and topological groupoids.

3. Invariant Markov operators

3.A. Markov chains and operators. Let X be a Borel space; a family $\pi = \{\pi_x\}$ of Borel probability measures on X indexed by points $x \in X$ is called Borel if for any non-negative Borel function f on X the function

$$\pi(f) : x \mapsto \langle \pi_x, f \rangle$$

is also Borel. Any such family determines a *Markov chain* on X with the transition probabilities π_x. As usual, we denote by $(X^{\mathbb{Z}_+}, \mathbf{P}_\theta)$ the space of *sample paths* $\overline{x} = (x_0, x_1, \dots)$ of this chain with the initial (σ-finite) distribution θ.

It is convenient to talk about a Markov chain in terms of the associated *Markov operator*

$$Pf = \pi(f) \; .$$

Its *dual operator* acts on the space of non-negative Borel measures on X by the formula

$$\theta P = \int \pi_x \, d\theta(x)$$

(this is a standard notation in the theory of Markov chains, e.g., see [Rev84]), so that

$$\langle \theta, Pf \rangle = \langle \theta P, f \rangle$$

for any non-negative Borel function f and measure θ (both sides are allowed to take infinite values as well). Then the one-dimensional distributions of the measure $\mathbf{P}_\theta$ in the path space are θP^n.

A measure (not necessarily finite!) θ on X is called *invariant* (or *stationary*) with respect to the operator P if $\theta = \theta P$ (or, equivalently, if the associated measure $\mathbf{P}_\theta$ in the path space is shift-invariant), *quasi-invariant* if θ and θP are equivalent, and *adapted* if the measure θP is absolutely continuous with respect to θ (for the lack of a well-established term; more legitimate if cumbersome candidates would be "sub-quasi-invariant", "quasi-excessive" or "null-preserved"). In the latter case the operator P also acts on the space $L^\infty(X, \theta)$. Alternatively, one can define a Markov operator directly as an operator on $L^\infty(X, \theta)$ [Fog69].

3.B. Invariant Markov operators.

Definition 3.1. A Markov operator on a Borel groupoid $\mathbf{G}$ is called *invariant* if its transition probabilities $\{\pi_{\mathbf{g}}\}$ satisfy the relation

$$\pi_{\mathbf{g}'\mathbf{g}} = \mathbf{g}'\pi_{\mathbf{g}} \qquad \forall\, (\mathbf{g}', \mathbf{g}) \in \mathbf{G}^{(2)} \; . \tag{3.2}$$

In other words, any transition probability $\pi_{\mathbf{g}}$ of an invariant Markov operator is concentrated on the corresponding fiber $\mathbf{G}^{\mathsf{t}(\mathbf{g})}$, i.e.,

$$\pi_{\mathbf{g}}(\mathbf{G}^{\mathsf{t}(\mathbf{g})}) = 1 \; , \tag{3.3}$$

and the relation (3.2) holds for any $\mathbf{g}' \in \mathbf{G}_{\mathsf{t}(g)}$. Thus, an invariant Markov operator P on $\mathbf{G}$ can be considered as a $\mathbf{G}$-invariant collection of Markov operators P_x on the fibers $\mathbf{G}^x$, $x \in \mathbf{G}^{(0)}$.

Proposition 3.4. *Any Borel system of probability measures on the fibers $\mathbf{G}^x$, $x \in \mathbf{G}^{(0)}$ of the target map $\mathsf{t} : \mathbf{G} \to \mathbf{G}^{(0)}$ uniquely extends to the system of transition probabilities of an invariant Markov operator on $\mathbf{G}$.*

Proof. We shall consider a system of probability measures π_x, $x \in \mathbf{G}^{(0)}$ on the fibers $\mathbf{G}^x$ as the transition probabilities from the associated points $\boldsymbol{\varepsilon}_x \in \mathbf{G}$. If we define

$$\pi_{\mathbf{g}} = \mathbf{g}\pi_{\mathsf{s}(\mathbf{g})} \qquad \forall\, \mathbf{g} \in \mathbf{G} \,, \tag{3.5}$$

then

$$\pi_{\mathbf{g}'\mathbf{g}} = (\mathbf{g}'\mathbf{g})\pi_{\mathsf{s}(\mathbf{g}'\mathbf{g})} = \mathbf{g}'\big(\mathbf{g}\pi_{\mathsf{s}(\mathbf{g})}\big) = \mathbf{g}'\pi_{\mathbf{g}} \qquad \forall\, (\mathbf{g}', \mathbf{g}) \in \mathbf{G}^{(2)} \,,$$

so that the system (3.5) is $\mathbf{G}$-invariant. □

In the measure theoretical setup, when talking about an invariant Markov operator P on a measured groupoid $(\mathbf{G}, \lambda, \mu)$ *we shall always assume that the measure $\lambda \star \mu$ is P-adapted*, which is equivalent to the measures λ^x from the Haar system being adapted with respect to the corresponding fiberwise Markov operators P_x for μ-a.e. $x \in \mathbf{G}^{(0)}$.

Note that we do not impose on invariant Markov operators P any conditions related to existence of a *P-invariant* measure.

3.C. Absolutely continuous transition probabilities. Given a Borel groupoid $\mathbf{G}$ with a Borel Haar system $\{\lambda^x\}$, we shall say that the transition probabilities $\{\pi_{\mathbf{g}}\}$ of an invariant Markov operator P are *absolutely continuous* if

$$\pi_{\mathbf{g}} \prec \lambda^{\mathsf{t}(\mathbf{g})} \qquad \forall\, \mathbf{g} \in \mathbf{G}$$

(we use the symbol $\prec$ to denote the absolute continuity of one measure with respect to another one). In view of Definition 3.1 and Proposition 3.4 this condition is equivalent to the absolute continuity just of the measures π_x, $x \in \mathbf{G}^{(0)}$ with respect to the corresponding measures λ^x. It is easy to see that the absolute continuity of the transition probabilities of an invariant Markov operator P on a measured groupoid $(\mathbf{G}, \lambda, \mu)$ implies that the measure $\lambda \star \mu$ is P-adapted. [Actually, it is sufficient to require absolute continuity of transition probabilities $\pi_{\mathbf{g}}$ just for $\lambda \star \mu$-a.e. $\mathbf{g} \in \mathbf{G}$, or, equivalently, just for μ-a.e. $\boldsymbol{\varepsilon}_x \cong x \in \mathbf{G}^{(0)}$.] Therefore, an invariant Markov operator with absolutely continuous transition probabilities acts on the space $L^\infty(\mathbf{G}, \lambda \star \mu)$.

Remark 3.6. Absolute continuity of transition probabilities is not necessary for the measure $\lambda \star \mu$ being adapted with respect to an invariant Markov operator P. The simplest example is provided by random walks on groups, see Example (i) below, in which case the Haar measure is adapted (even invariant) with respect to any random walk on the group.

Remark 3.7. As we have seen in Proposition 3.4, invariant Markov operators on a groupoid $\mathbf{G}$ are in one-to-one correspondence with systems of probability measures on the fibers of the target map $\mathsf{t} : \mathbf{G} \to \mathbf{G}^{(0)}$. The product of invariant Markov operators corresponds then to the usual convolution in the space of such systems, or, for operators with absolutely continuous transition probabilities, in the space of densities of these measures with respect to a Haar system, see [Ren80], [ADR00] for the definition of convolution on groupoids.

3.D. Examples of invariant Markov operators. Markov operators satisfying reasonable "homogeneity conditions" can, as a rule, be interpreted as invariant Markov operators on appropriate groupoids.

(i) *Random walks on groups.* According to the classical definition, a (right) *random walk* on a locally compact group G determined by a probability measure π on G is the Markov chain on G with the transition probabilities $\pi_g = g\pi$ which are equivariant with respect to the action of the group on itself on the left. Its transition operator P is an invariant Markov operator on the associated groupoid $\mathbf{G}$ (see example (i) in Section **1.B**). For introducing a structure of a measured groupoid on $\mathbf{G}$ it is sufficient to take a left

Haar measure on the group G (which can be considered as a "Haar system", since the object space $\mathbf{G}^{(0)}$ is a singleton), and absolute continuity of transition probabilities of the operator P is equivalent just to absolute continuity of the measure π with respect to λ.

(ii) *Markov chains on equivalence relations and foliations.* An equivalence relation R on a standard Borel space X is called *standard* if it is a Borel subset of $X \times X$, and it is called *countable* if the equivalence class (the *leaf*) $[x] = R(x) = \{y : (x, y) \in R\}$ of any point $x \in X$ is at most countable. A standard countable equivalence relation is also called *discrete*. A standard equivalence relation R is called *non-singular* with respect to a Borel probability measure μ on X (or, equivalently, the measure μ is *quasi-invariant* with respect to R) if for any subset $A \subset X$ with $\mu(A) = 0$ its *saturation* $[A] = \bigcup_{x\in A}[x]$ also has measure 0, see [FM77].

Let $\mathbf{G}$ be the groupoid associated with the equivalence relation R, see Section **1.B**. Then the counting measures on the sets $\mathbf{G}^x = x \times [x]$ provide a Haar system λ for $\mathbf{G}$, and the R-quasi-invariance of a measure μ on X in the above sense is equivalent to its quasi-invariance with respect to the Haar system λ, so that $(\mathbf{G}, \lambda, \mu)$ is a measured groupoid.

A *random walk along the classes of the equivalence relation* R [Kai98] is determined by a family of probability measures π_x, $x \in X$ concentrated on the corresponding classes $[x]$, which is Borel in the sense that the function $(x, y) \mapsto \pi_x(y)$ on R is Borel. The associated invariant Markov operator on the groupoid $\mathbf{G}$ has the transition probabilities

$$p\big((x, y), (x, z)\big) = \pi_y(z) ,$$

which are obviously absolutely continuous with respect to the Haar system.

The same construction is applicable to the leafwise diffusion processes on *measured Riemannian foliations* as well. In this case the role of the Haar system can be played by the leafwise Riemannian volumes, see [KL01].

The next 3 examples are obtained from random walks on groups by further "randomizing" them in various ways.

(iii) *Random walks with internal degrees of freedom (RWIDF).* In this model first introduced by Krámli and Szász [KS83] the random walk on a countable group G is driven by a Markov chain on a space X which describes the internal or hidden "degrees of freedom" of the observed process on G. The state space of the RWIDF is the product $\widetilde{X} = X \times G$, and the transition probabilities $\pi_{(x,g)} = g\pi_x$ are equivariant with respect to the action of the group on itself, where π_x are probability measures on $\widetilde{X}$ indexed by points $x \in X$. In particular, when X is a singleton this is just the usual random walk on G. In the terminology from [Kai95] the Markov operator of RWIDF is a *covering operator* with the *deck group* G. The class of covering Markov operators (in other words, of RWIDF) includes, in particular, the Brownian motion on covering Riemannian manifolds.

Let us define the corresponding groupoid $\mathbf{G}$ as the product of the groupoid associated with the full equivalence relation on the set X and the groupoid associated with the group G (see Section **1.B**). Namely, its set of morphisms and set of objects are

$$\mathbf{G} = X \times X \times G , \qquad \mathbf{G}^{(0)} = X , \tag{3.8}$$

respectively, with obvious definitions of the structure maps. Then the random walk with internal degrees of freedom gives rise to an invariant Markov operator P on the groupoid $\mathbf{G}$, for which the corresponding probability measures on the fibers $\mathbf{G}^x$ of the target map (see Proposition 3.4) are precisely the measures π_x.

Any measure μ on $X = \mathbf{G}^{(0)}$ is obviously quasi-invariant with respect to the Haar system λ on G consisting of products of μ and the counting measure λ_G on G, which

gives a structure of a measured groupoid on $\mathbf{G}$. Absolute continuity of the transition probabilities of the operator P is then equivalent to absolute continuity of the measures π_x with respect to the product $\mu \otimes \lambda_G$, or just to absolute continuity of the transition probabilities of the quotient chain on X with respect to the measure μ.

Under suitable assumptions one can pass in the above construction from countable groups G to general locally compact groups. However, the situation when the action of G on the state space $\widetilde{X}$ is not free does not readily fit into this scheme. One has either to pass to the more general construction of invariant operators on homogeneous spaces of groupoids or to lift the Markov operator from the homogeneous space to an appropriately defined invariant operator on the groupoid, see [KW02], [SCW02]

(iv) *Random walks with random transition probabilities (RWRTP).* This is a specialization of the above model of RWIDF to the situation when the quotient chain on X is deterministic, i.e., consists in moving along the orbits of a certain transformation T. Then for any $x \in X$ the corresponding transition probability π_x on $X \times G$ is concentrated on the set $\{Tx\} \times G$, i.e., can be considered as a probability measure on G. Let us endow the space X with a T-invariant probability measure μ. The associated RWRTP consists in picking up randomly (with distribution μ) a point $x \in X$ and performing the first step of the random walk on G with the jump distribution π_x, then at the next moment of time passing to the point Tx and performing the next step on G with the jump distribution π_{Tx}, etc. Therefore, the observed "random" Markov chain on G is homogeneous in space, but not in time, its transition probabilities at time n being $p_n(g, gh) = \pi_{T^n x}(h)$, see [KKR02].

In order to make the transition probabilities of the invariant Markov operator arising in this model absolutely continuous one has to pass from the "big" groupoid $\mathbf{G}$ (3.8) defined in the previous example to the subgroupoid $\mathbf{G}' \subset \mathbf{G}$ generated by the supports of the transition probabilities. In this case $\mathbf{G}'$ is the product of the groupoid of the orbit equivalence relation R_T of the transformation T and the group G. Its set of morphisms is

$$\mathbf{G}' = \{(x, T^k x, g) : x \in X, g \in G, k \in \mathbb{Z}\} \subset \mathbf{G} \ .$$

and the set objects is the same as for $\mathbf{G}$, i.e., X. Then the products of counting measures on the classes of R_T and the counting measure on the group G will provide a Haar system on $\mathbf{G}'$, and the transition probabilities of the arising invariant Markov operator on $\mathbf{G}'$ will obviously be absolutely continuous with respect to this Haar system.

(v) *Random walks in random environment (RWRE).* This (historically, the first model of a randomization of the usual random walk on a group) is yet another specialization of RWIDF in a sense opposite to RWRTP: the arising random Markov chains on the group G are homogeneous in time, but not in space. It was introduced by Solomon [Sol75] and profoundly studied later (mostly for abelian groups with few exceptions, though).

Denote by $\mathcal{M}(G)$ the space of (infinite) configurations $\omega = \{\omega_g\}_{g \in G}$ on G with the values in the space $\mathcal{P}(G)$ of probability measures on G. The group G acts on $\mathcal{M}(G)$ by translations $(g\omega)_{g'} = \omega_{g^{-1}g'}$. The space $\mathcal{M}(G)$ can be identified with the space of all Markov operators on G: the transition probabilities of the Markov operator P_ω determined by ω are

$$p_\omega(g, gh) = \omega_g(h) = (g^{-1}\omega)_e(h) = p_{g^{-1}\omega}(e, h) \ ,$$

i.e., ω_g is the distribution of the "right increment" h at the point g.

Let now X be a G-space with a G-quasi-invariant measure μ (the *space of environments*) endowed with a map $x \mapsto \pi_x \in \mathcal{P}(G)$. By equivariance this map can be extended to a

map from X to $\mathcal{M}(G)$: we shall consider π_x as the value of the configuration $\omega(x)$ at the group identity e and then put

$$\omega(x) = \{\pi_{g^{-1}x}\}_{g\in G} \in \mathcal{M}(G) .$$

The associated RWRE consists then in picking up randomly (with distribution μ) a point $x \in X$ and running on G the "random" Markov chain with the transition probabilities $p_x(g, gh) = \pi_{g^{-1}x}(h)$ (actually, instead of the space (X, μ) one could consider directly the space $\mathcal{M}(G)$ endowed with the image of the measure μ under the map $x \mapsto \omega(x)$).

This model gives rise to the covering Markov chain (RWIDF) on the space $\widetilde{X} = X \times G$ with the transition probabilities $\widetilde{\pi}_{(x,g)} = g\widetilde{\pi}_x$, where $\widetilde{\pi}_x$ are the images of the measures π_x under the map $h \mapsto (h^{-1}x, h)$. Sample paths (x_n, g_n) of this chain have the following interpretation: (g_n) is a sample path of the Markov chain run in the environment $g_0^{-1}x_0$, whereas (x_n) is a sample path of the Markov chain in the space of environments corresponding to the so-called "moving coordinate system" (in which the Markov particle is always situated at the group identity, whereas the environment around it changes).

In the same way as in the case of RWRTP above, in order to make the transition probabilities of the arising invariant Markov operator absolutely continuous one has to pass from the groupoid $\mathbf{G}$ (3.8) to a subgroupoid $\mathbf{G}' \subset \mathbf{G}$ generated by the supports of transition probabilities. In the case of RWRE

$$\mathbf{G}' = \{(x, g^{-1}x, g) : x \in X, g \in G\} ,$$

i.e., this is precisely the groupoid determined by the action of G on X.

4. The Liouville property

4.A. **Bounded harmonic functions and the Liouville property.** The classical Liouville theorem asserts absence of bounded harmonic functions on the Euclidean space. The notion of a harmonic function (based on the mean value property) can in fact be defined for an arbitrary Markov chain. Namely, a Borel function f on the state space X of a Markov chain is called harmonic if $f = Pf$, where P is the transition operator of the Markov chain. In the measure theoretical setup, given a P-adapted measure m on X, the classes (mod 0) of P-harmonic functions form a closed subspace $H^\infty(X, m, P) \subset L^\infty(X, m)$. The operator P is then called *Liouville* (with respect to the measure m) if the space $H^\infty(X, m, P)$ consists of constant functions only.

The link between the Liouville property and amenability is based on the so-called *0–2 laws* for Markov operators due to Derriennic [Der76] (also see [Kai92]), which assert that absence of non-constant bounded harmonic functions is equivalent to asymptotic independence of n-step transition probabilities of initial states. This is precisely what is needed for constructing approximatively invariant sequences of probability measures from condition (2.1). Yet another, less constructive, way of connecting the Liouville property with amenability consists in the observation that any fixed *measure-linear mean* on $\mathbb{Z}_+$ when applied to the values of an arbitrary bounded measurable function on the state space of a Liouville Markov operator along the sample paths of the associated Markov chain provides a projection onto the space of constants which is invariant with respect to all the symmetries of the operator [CFW81], [LS84], [KF98].

In the case of an invariant Markov operator P on a measured groupoid $(\mathbf{G}, \lambda, \mu)$ any measured function on $\mathbf{G}$ which is constant on a.e. fiber $\mathbf{G}^x$ of the target map $\mathsf{t} : \mathbf{G} \to \mathbf{G}^{(0)}$ is necessarily harmonic by formula (3.3), so that it makes sense to talk about the Liouville property for each of the operators P_x on the fibers $\mathbf{G}^x$.

4.B. Fiberwise Liouville operators.

Definition 4.1. An invariant Markov operator P on a measured groupoid $(\mathbf{G}, \lambda, \mu)$ is called *fiberwise Liouville* if for μ-a.e. $x \in \mathbf{G}^{(0)}$ the operator $P_x : L^\infty(\mathbf{G}^x, \lambda^x) \hookleftarrow$ is Liouville. A measured groupoid $(\mathbf{G}, \lambda, \mu)$ is called *Liouville* if it carries a fiberwise Liouville invariant Markov operator.

Theorem 4.2. *Any Liouville measured groupoid is amenable.*

Proof. By one of the 0–2 laws a Markov operator $P : L^\infty(X, m) \hookleftarrow$ is Liouville if and only if for any two probability measures $\theta_1, \theta_2 \prec m$

$$\left\| \frac{1}{n+1} \sum_{k=0}^{n} (\theta_1 - \theta_2) P^k \right\| \underset{n\to\infty}{\longrightarrow} 0 \ . \tag{4.3}$$

Note that if the transition probabilities π_x of the operator P are absolutely continuous with respect to the measure m, then it is sufficient to consider in formula (4.3) just the δ-measures $\theta_i = \delta_{x_i}$, $x_i \in X$ $(i = 1, 2)$.

Let now $P : L^\infty(\mathbf{G}, \lambda \star \mu) \hookleftarrow$ be a fiberwise Liouville invariant Markov operator. Take a measurable system of absolutely continuous probability measures $\theta^x \prec \lambda^x$ on the fibers $\mathbf{G}^x$ of the target map, and let

$$\theta^x_n = \frac{1}{n+1} \sum_{k=0}^{n} \theta^x P^k \prec \lambda^x \ .$$

By $\mathbf{G}$-invariance of the operator P for any $\mathbf{g} \in \mathbf{G}$

$$\left\| \mathbf{g}\theta_n^{\mathsf{s}(\mathbf{g})} - \theta_n^{\mathsf{t}(\mathbf{g})} \right\| = \left\| \frac{1}{n+1} \sum_{k=0}^{n} \left(\mathbf{g}\theta^{\mathsf{s}(\mathbf{g})} - \theta^{\mathsf{t}(\mathbf{g})} \right) P^k \right\| \ , \tag{4.4}$$

where $\mathbf{g}\theta^{\mathsf{s}(\mathbf{g})}$ and $\theta^{\mathsf{t}(\mathbf{g})}$ are probability measures on the fiber $\mathbf{G}^{\mathsf{t}(\mathbf{g})}$ absolutely continuous with respect to $\lambda^{\mathsf{t}(\mathbf{g})}$. Since the fiberwise operators are a.e. Liouville, the 0–2 law (4.3) implies a.e. convergence of (4.4) to zero, and therefore, in view of formula (2.1), amenability of the groupoid $(\mathbf{G}, \lambda, \mu)$. □

Remark 4.5. Another (non-constructive) proof of Theorem 4.2 can be given by using a measure-linear mean along the sample paths of the Markov chain to obtain an invariant mean from $L^\infty(\mathbf{G})$ to $L^\infty(\mathbf{G}^{(0)})$, see the proof of Theorem 5.2 below.

4.C. Applications and examples. Particular cases of Theorem 4.2 were earlier established for groups [Aze70], [Fur73], equivalence relations and foliations [CFW81], for the Brownian motion on covering manifolds [LS84], for general covering Markov operators [Kai95] as well as for various models of randomization of the usual random walk on a discrete group, see [Sun87], [KKR02].

It is plausible that the converse may also be true:

Conjecture 4.6. *Any amenable measured groupoid is Liouville.*

This is known to be the case for groups [Ros81], [KV83] (any amenable group carries a random walk with the trivial Poisson boundary; it had been previously conjectured by Furstenberg [Fur73]). The proof of Conjecture 4.6 in full generality should presumably follow the same strategy of constructing a Liouville operator from Følner sets on the groupoid. Other known particular cases are the groupoids associated with discrete equivalence relations (in view of the Connes–Feldman–Weiss theorem it is the orbit equivalence relation of a $\mathbb{Z}$-action [CFW81]) and with group actions (in a somewhat weaker form,

though; by [EG93], [AEG94] any amenable measure class preserving action of a locally compact group G can be realized as the action on the Poisson boundary of an appropriate G-invariant operator, see below Section **5.D**; proving Conjecture 4.6 for group actions would provide an alternative link between amenability and the Poisson boundary). Note that the proof of hyperfiniteness of amenable equivalence relations in [CFW81] (also see [Kai97]) is non-constructive and uses the Zorn Lemma; looking for a more direct Følner type argument would provide an insight into the general case.

Remark 4.7. There is a generalization to group extensions of the aforementioned existence of a Liouville random walk on any amenable locally group. Namely, if G is a locally compact group, and H its closed normal subgroup, then H is amenable if and only if for any Borel probability measure π' on the quotient group $G' = G/H$ there exists a Borel lift π to G such that the Poisson boundaries of the random walks (G, π) and (G', π') are canonically isomorphic [Kai02]. In spite of having the same spirit as Conjecture 4.6, this result does not seem to have an obvious interpretation in groupoid terms.

4.D. **Amenability of group actions.** We shall now give a specialization of Theorem 4.2 to the particular case of (groupoids associated with) group actions. In this situation the amenability of a measure class preserving action of a locally compact group G on a measure space (X, μ) is equivalent to existence of a sequence of measurable maps θ_n from the action space X to the space $\mathcal{P}(G)$ of probability measures on G which are *approximatively equivariant* in the sense that $\|g\theta_n(x) - \theta_n(gx)\| \to 0$ weakly, cf. formula (2.1). Recall that a Borel G-space S is called proper if it carries a Borel G-invariant system of probability measures on G [ADR00, Definition 2.1.2] (for continuous actions on locally compact spaces it follows prom properness in the usual sense). Such a system allows one to lift any probability measure from S to G in an equivariant measurable way. We shall need this property in the measure theoretical setup and say that an action of a locally compact group G on a measure space (S, m) is *proper* if there exists a measurable equivariant map from the space of probability measures on S absolutely continuous with respect to m to the space of probability measures on G.

Theorem 4.8. *Let G be a locally compact group acting measure preserving and properly on a measure space (S, m), and let $P : L^\infty(S, m) \hookleftarrow$ be a G-invariant Markov operator. Suppose that the group G also has a measure class preserving action on another space (X, μ), and there is a measurable G-equivariant map assigning to points $x \in X$ projective classes of minimal positive Borel P-harmonic functions φ_x. Then the action of G on (X, μ) is amenable.*

Proof. Recall that any positive P-harmonic function φ determines a new Markov operator $P^\varphi f = P(\varphi f)/\varphi$ on $L^\infty(S, m)$ called the *Doob transform* of the operator P determined by the function φ [Rev84]. In other words, the transition probabilities of the Doob transform are determined by the formula

$$\frac{d\pi_s^\varphi}{d\pi_s}(t) = \frac{\varphi(t)}{\varphi(s)} , \tag{4.9}$$

where π_s are the transition probabilities of the operator P. The Doob transform remains the same if the function φ is multiplied by a positive constant, so that actually it depends just on the projective class of φ, or, in other words, on the multiplicative cocycle $(s, t) \mapsto \varphi(t)/\varphi(s)$. One can easily see that the minimality of the function φ is equivalent to the Liouville property for the operator P^φ, which provides the required link between minimality and the Liouville property.

Strictly speaking, the situation considered in Theorem 4.8 is slightly different from the setup of Theorem 4.2 as here we arrive at an invariant Markov operator on a homogeneous space of the action groupoid rather than just on the groupoid itself (cf. the discussion at the end of Example (iii) in Section **3.D**), which is why we shall briefly outline the rest of the proof.

Let us fix a probability measure $\theta \prec m$ on S, and consider the family

$$\theta_n^x = \frac{1}{n+1} \sum_{k=0}^{n} \theta (P^x)^k \prec m$$

of probability measures on S parameterized by points $x \in X$, where $P^x = P^{\varphi_x}$ are the Doob transforms associated with the functions φ_x. Then, in the same way as in the proof of Theorem 4.2, the systems θ_n^x are approximatively equivariant, i.e., for any $g \in G$

$$\| g\theta_n^x - \theta_n^{gx} \| \to 0 \ .$$

Since the action of G on (S, m) is proper, the measures θ_n^x can now be equivariantly and measurably lifted from S to G to provide an approximatively equivariant sequence of measurable maps from X to $\mathcal{P}(G)$. □

Remark 4.10. Theorem 4.8 and its topological analogue Theorem 6.3 carry over *verbatim* to the situation when φ_x are λ-*harmonic* minimal functions for a certain fixed eigenvalue $\lambda > 0$ (i.e., $P\varphi_x = \lambda\varphi_x$). In this case the definition of the Doob transform has to be modified by dividing the right-hand side of formula (4.9) by λ.

Remark 4.11. The metric characterization of minimal harmonic functions resulting from applying the 0–2 law to the corresponding Doob transform first appeared in author's paper [Kai83].

Remark 4.12. The properness assumption in Theorem 4.8 is essential. In a sense, it says that the measurable structures on the group G and on the G-space S agree. For instance, take a free dense subgroup F of a compact group K, and consider on K the random walk determined by a transition probability π supported by the generating set of F. Then the Poisson boundary of the associated transition operator on the space (K, λ) (where λ is the Haar measure on K) is trivial (there are no measurable π-harmonic functions on K, which follows from the ergodicity of the action of F and the fact that λ is a finite stationary measure of this random walk), but the action of the free group on a singleton is not amenable. This example illustrates the importance of the choice of an ambient measurable structure in the definition of the Poisson boundary.

Remark 4.13. In the Borel setup, when the map $x \mapsto \varphi_x$ is well-defined for *all* points $x \in X$, the argument from the proof of Theorem 4.8 is applicable to an arbitrary quasi-invariant measure μ on X to provide the *measurewise amenability* [ADR00] (other terms: *universal amenability* [Ada96], *measure-amenability* [JKL02]) of the action of G on X; cf. Remark 6.4 below).

See Section **6.C** for examples of application of Theorem 4.8 (or, rather, of its topological refinement Theorem 6.3).

5. The Poisson extension

5.A. **The Poisson boundary.** Let m be a P-adapted Borel σ-finite measure of a Markov operator P on a state space X, so that the operator P acts on the space $L^\infty(X, m)$. We shall assume that (X, m) is a *Lebesgue space* (in particular, this is always the case when

X is *Polish*). Then the associated path space $(X^{\mathbb{Z}_+}, \mathbf{P}_m)$ is also a Lebesgue space, and the space $\Gamma = \Gamma(X, m, P)$ of the ergodic components of the time shift T in the path space is called the *Poisson boundary* of the operator P. By definition, there is a canonical measurable projection **bnd** from the path space onto the Poisson boundary constant along the orbits of the time shift, and the Poisson boundary is the maximal quotient of the path space with this property. The Poisson boundary (which is defined in the measure theoretical category only!) is endowed with the *harmonic measure class* $[\nu_m] = \mathbf{bnd}\,[\mathbf{P}_m]$. For convenience we shall fix a probability measure $\nu \in [\nu_m]$. For instance, one can take $\nu = \mathbf{bnd}\,\mathbf{P}_\theta$ for any probability measure θ on the state space equivalent to m. For any initial distribution $\theta \prec m$ the associated *harmonic measure* $\nu_\theta = \mathbf{bnd}\,\mathbf{P}_\theta$ is absolutely continuous with respect to the harmonic measure class. If the operator P has absolutely continuous transition probabilities then the individual harmonic measures ν_x (corresponding to the initial distributions δ_x, $x \in X$) are also well-defined and absolutely continuous with respect to the harmonic measure class.

The space of (classes mod 0 of) bounded harmonic functions $H^\infty(X, m, P)$ of the operator P is canonically isomorphic to the space $L^\infty(\Gamma, [\nu_m])$. For Markov operators with absolutely continuous transition probabilities this isomorphism is established by the *Poisson formula*

$$f(x) = \langle \nu_x, \widehat{f} \rangle = \int \widehat{f}(\widehat{\gamma})\, \Pi(x, \gamma)\, d\nu(\gamma) ,$$

where $\Pi(x, \gamma) = d\nu_x / d\nu(\gamma)$ is the *Poisson kernel*. For $[\nu_m]$-a.e. point $\gamma \in \Gamma$ the function $\Pi(\cdot, \gamma)$ is a minimal P-harmonic function. Actually, the Poisson formula can be given sense in full generality as well by defining the individual harmonic measures (not necessarily absolutely continuous with respect to the harmonic measure class anymore!) by using Rokhlin's theorem on conditional probabilities in Lebesgue spaces (cf. the proof of Theorem 5.2 below).

See [Kai92] and the references therein for more detailed information on the Poisson boundary of Markov operators.

5.B. **The Poisson boundary of invariant operators.** Let now P be an invariant Markov operator on a measured groupoid $(\mathbf{G}, \lambda, \mu)$, so that the measure $\lambda \star \mu$ is P-adapted. We denote sample paths from $\mathbf{G}^{\mathbb{Z}_+}$ by $\overline{\mathbf{g}} = (\mathbf{g}_0, \mathbf{g}_1, \dots)$. Since the target map t is constant along the sample paths of an invariant Markov operator by formula (3.3), it can be extended to a projection map (also denoted t) from the path space to $\mathbf{G}^{(0)}$. Therefore, the action of $\mathbf{G}$ on itself extends to a coordinate-wise action of $\mathbf{G}$ on the path space by the formula

$$\mathbf{g}\overline{\mathbf{g}} = (\mathbf{g}\mathbf{g}_0, \mathbf{g}\mathbf{g}_1, \dots) , \qquad \mathsf{s}(\mathbf{g}) = \mathsf{t}(\overline{\mathbf{g}}) ,$$

where $\mathbf{g}_n$ are the coordinates of the sample path $\overline{\mathbf{g}}$. Since the measure $\lambda \star \mu$ on $\mathbf{G}$ is quasi-invariant with respect to $(\mathbf{G}, \lambda)$ (see Section **1.D**), the associated measure $\mathbf{P}_{\lambda\star\mu}$ on the path space is also $(\mathbf{G}, \lambda)$-quasi-invariant. Further, since the action of the time shift on the path space commutes with the action of $\mathbf{G}$, we obtain that this action descends to the Poisson boundary Γ of the operator P, and that the harmonic measure class on Γ is quasi-invariant with respect to this action. The target map descends to Γ from the path space; its fibers are the Poisson boundaries of the fiberwise Markov operators $P_x : L^\infty(\mathbf{G}^x, \lambda^x) \hookleftarrow$ (cf. [KKR02, Proposition 1.11]).

5.C. The Poisson extension.

Definition 5.1. Given an invariant Markov operator P on a measured groupoid $(\mathbf{G}, \lambda, \mu)$, we shall call the *Poisson extension* the measured groupoid $\widetilde{\mathbf{G}} = \mathbf{G} \ltimes \Gamma$ associated with the action of $\mathbf{G}$ on the Poisson boundary Γ of the operator P.

The following result is a generalization of Theorem 4.2 (if an invariant operator is fiberwise Liouville, then its Poisson extension is just the original groupoid).

Theorem 5.2. *The Poisson extension of any invariant Markov operator on a measured groupoid is amenable.*

Proof. One way of proving Theorem 5.2 is to deduce it from Theorem 4.2 by showing that the groupoid $\widetilde{\mathbf{G}}$ carries a natural fiberwise Liouville invariant Markov operator. Such an operator is obtained by conditioning the original operator P by the points of the Poisson boundary. This is easily done in the case when P has absolutely continuous transition probabilities. Namely, a.e. point $\gamma \in \Gamma$ determines the conditional Markov operator P^γ on the fiber $\mathbf{G}^{\mathsf{t}(\gamma)}$, for which the measure $\lambda^{\mathsf{t}(\gamma)}$ is adapted. These are the Doob transforms corresponding to the fiberwise Poisson kernels. In other words, the transition probabilities of P^γ satisfy the relation (4.9)

$$\frac{d\pi^\gamma_{\mathbf{g}}}{d\pi_{\mathbf{g}}}(\mathbf{g}') = \frac{d\nu_{\mathbf{g}'}}{d\nu_{\mathbf{g}}}(\gamma) \ ,$$

where $\nu_{\mathbf{g}}$ are the harmonic measures on the Poisson boundary (it is here that we need the absolute continuity of the transition probabilities which guarantees existence of fiberwise Poisson kernels). Since the Poisson kernel consists of minimal harmonic functions, the operators $P^\gamma : L^\infty\left(\mathbf{G}^{\mathsf{t}(\gamma)}, \lambda^{\mathsf{t}(\gamma)}\right) \hookleftarrow$ are Liouville.

It remains to notice that the conditional operators P^γ can be interpreted as fiberwise operators of an invariant Markov operator $\widetilde{P}$ on the Poisson extension $\widetilde{\mathbf{G}}$ and to apply Theorem 4.2. Indeed, the elements of $\widetilde{\mathbf{G}} = \mathbf{G} \ltimes \Gamma$ are the triples $(\gamma, \mathbf{g}, \mathbf{g}^{-1}\gamma)$ (see Section **1.C**). Fixing the target $\gamma = \mathsf{t}(\gamma, \mathbf{g}, \mathbf{g}^{-1}\gamma)$ (which corresponds to conditioning by γ as an element of the Poisson boundary) we may identify the corresponding fiber $\widetilde{\mathbf{G}}^\gamma$ with $\mathbf{G}^{\mathsf{t}(\gamma)}$ by the formula $(\gamma, \mathbf{g}, \mathbf{g}^{-1}\gamma) \leftrightarrow \mathbf{g}$ and then define the transition probabilities

$$\widetilde{\pi}_{(\gamma,\mathbf{g},\mathbf{g}^{-1}\gamma)} = \pi^\gamma_{\mathbf{g}} \ ,$$

which are clearly $\widetilde{\mathbf{G}}$-invariant in view of $\mathbf{G}$-invariance of the probabilities $\pi_{\mathbf{g}}$, cf. Example (v) from Section **3.D**.

Let us sketch how this argument can be carried over to the general case. Since the measure $\lambda \star \mu$ is P-adapted, for the associated measure in the path space $T\mathbf{P}_{\lambda\star\mu} \prec \mathbf{P}_{\lambda\star\mu}$ (here T is the time shift). If θ is a probability measure equivalent to $\lambda \star \mu$, then clearly

$$T\mathbf{P}_\theta \prec \mathbf{P}_\theta \ . \tag{5.3}$$

Denote by $\mathbf{P}^\gamma_\theta$, $\gamma \in \Gamma$ the T-ergodic components of the measure $\mathbf{P}_\theta$, i.e., its conditional measures with respect to the Poisson boundary. By definition of the Poisson boundary these measures are Markov and, by (5.3), $T\mathbf{P}^\gamma_\theta \prec \mathbf{P}^\gamma_\theta$ for a.e. $\gamma \in \Gamma$. Therefore the one-dimensional distributions θ^γ of the measures $\mathbf{P}^\gamma_\theta$ at time 0 are adapted with respect to the corresponding conditional Markov operators P^γ (although these measures may well be singular with respect to the Haar measures $\lambda^{\mathsf{t}(\gamma)}$). The rest of the argument then goes in the same way as in the absolutely continuous case.

However, it is more convenient to use for proving Theorem 5.2 another approach based on using *measure-linear means*. Mokobodzki (see [Mey73], [Fis87]) proved that there exists an invariant mean ξ on $\mathbb{Z}_+$ (called *measure-linear* or *medial*) with the following remarkable property: it is *universally measurable* as a map from the product space $[-1,1]^{\mathbb{Z}_+}$ to $[-1,1]$, i.e., the integral in the right hand side below is well-defined for any Borel probability measure μ on $[-1,1]^{\mathbb{Z}_+}$, and

$$\xi\left\{\int \boldsymbol{a}\, d\mu(\boldsymbol{a})\right\} = \int \xi\{\boldsymbol{a}\}\, d\mu(\boldsymbol{a}) .$$

In view of the definition from Section **2.B** for proving amenability of the groupoid $\widetilde{\mathbf{G}}$ we have to construct an invariant mean $\mathbf{\Pi} : L^\infty(\widetilde{\mathbf{G}}) \to L^\infty(\Gamma)$ (recall that the space of objects of $\widetilde{\mathbf{G}} = \mathbf{G} \ltimes \Gamma$ is the Poisson boundary Γ). As above we shall parameterize $\widetilde{\mathbf{G}}$ by the map

$$(\mathbf{g},\gamma) \leftrightarrow (\gamma, \mathbf{g}, \mathbf{g}^{-1}\gamma) , \qquad \mathbf{g}\in\mathbf{G},\ \gamma\in\Gamma , \tag{5.4}$$

where $\mathsf{t}(\mathbf{g}) = \mathsf{t}(\gamma)$. Then the target map $\widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}^0$ is just $(\mathbf{g},\gamma)\mapsto\gamma$, and the measure class on $\widetilde{\mathbf{G}}$ with respect to which we consider the space $L^\infty(\widetilde{\mathbf{G}})$ is $d\lambda^{\mathsf{t}(\gamma)}(\mathbf{g})\, d[\nu](\gamma)$, where $[\nu]$ is the harmonic measure class on Γ corresponding to the initial distribution $\lambda\star\mu$ on $\mathbf{G}$. In the coordinates (5.4) the left action of $\widetilde{\mathbf{G}}$ on itself coincides just with the diagonal action of $\mathbf{G}$

$$\mathbf{h}(\mathbf{g},\gamma) \mapsto (\mathbf{hg}, \mathbf{h}\gamma) , \qquad \mathsf{s}(\mathbf{h}) = \mathsf{t}(\mathbf{g}) = \mathsf{t}(\gamma) .$$

Let us fix a reference system of probability measures $\rho = \{\rho^x\}$ on the fibers of the target map of $\mathbf{G}$ equivalent to the Haar system λ and put for any $F\in L^\infty(\widetilde{\mathbf{G}})$ and any sample path $\overline{\mathbf{g}} = (\mathbf{g}_n)$ on $\mathbf{G}$

$$\overline{\mathbf{\Pi}}F(\overline{\mathbf{g}}) = \xi\left\{\int F(\mathbf{g}_n\mathbf{h}, \mathbf{bnd}\,\overline{\mathbf{g}})\, d\rho^{\mathsf{s}(\mathbf{g}_n)}(\mathbf{h})\right\} .$$

Note that *a priori* the images of the measure $\mathbf{P}_{\lambda\star\mu}$ under the maps $\overline{\mathbf{g}} \mapsto (\mathbf{g}_n, \mathbf{bnd}\,\overline{\mathbf{g}})$ may well be singular with respect to the quasi-invariant measure class on $\widetilde{\mathbf{G}}$ (we keep using the coordinates (5.4)), however the additional integration with respect to the measures ρ^x guarantees that $\overline{\mathbf{\Pi}}F$ is well-defined as an element of $L^\infty(\mathbf{G}^{\mathbb{Z}_+}, \mathbf{P}_{\lambda\star\mu})$. Since ξ is a mean, the function $\overline{\mathbf{\Pi}}F$ is shift invariant, so that it descends to a measurable function

$$\mathbf{\Pi}F(\mathbf{bnd}\,\overline{\mathbf{g}}) = \overline{\mathbf{\Pi}}F(\overline{\mathbf{g}})$$

on the Poisson boundary. Since ξ is measure-linear, $\mathbf{\Pi}$ is a mean from $L^\infty(\widetilde{\mathbf{G}})$ to $L^\infty(\Gamma) \cong L^\infty(\widetilde{\mathbf{G}}^0)$. Finally, $\widetilde{\mathbf{G}}$-invariance of $\mathbf{\Pi}$ obviously follows from its definition. □

5.D. Examples. Theorem 5.2 has been earlier proved in the following particular cases.

For the groupoid $\mathbf{G}$ associated with a locally compact group G and the invariant Markov operator determined by a random walk (G,π) (see Section **3.D**) amenability of the Poisson extension amounts to ergodicity of the action of G on the Poisson boundary of the associated Markov operator on the space $L^\infty(G,\lambda)$ (where λ is the Haar measure on G). It was proved by Zimmer [Zim78] for an arbitrary measure π on G. Note that the formulation of Theorem 5.1 in [Zim78] contains the requirement that the measure π be *étalée* or *spread out* (i.e., have a convolution power non-singular with respect to the Haar measure). The reason is that there he used a definition of the Poisson boundary which differs from ours; it was supposed to represent *all* bounded harmonic functions rather then just classes mod 0 from $L^\infty(G,\lambda)$. However, Zimmer essentially deals precisely with the Poisson boundary

in our sense, and proves its amenability without any further assumptions on the measure π in his Theorem 5.2.

Zimmer used the fact that the increments of a random walk on a group form a stationary sequence. As it was pointed out in [CW89], his method does not seem to work in the non-stationary case. Connes and Woods [CW89] proved amenability of the action of a locally compact group G on the Poisson boundary for the so-called *matrix-valued random walks* on G which are G-invariant Markov chains on the product of G by a certain countable set subject to some additional assumptions on transition probabilities. Note that actually what they call the Poisson boundary is rather the *tail boundary* (the quotient of the path space by the *synchronous* asymptotic equivalence relation). Although it coincides with our Poisson boundary (the quotient of the path space by the *asynchronous* asymptotic equivalence relation) for matrix-valued random walks in the sense of [CW89], in general they may differ (the Poisson boundary being a quotient of the tail boundary, so that amenability of the action on the former is stronger than on the latter), for instance, see [Kai92], [Jaw95].

Elliott and Giordano [EG93] for discrete groups and Adams, Elliott, Giordano [AEG94] in the general case later proved that in fact any measure class preserving amenable action of a second countable locally compact group can be presented as its action on the Poisson boundary of an appropriately defined matrix-valued random walk.

6. Topological amenability

6.A. **Topological Liouville property.** Theorem 4.2 has an analogue in the topological category (see Section **2.C** for a definition of topological amenability) which can be proved along the same lines by using the 0–2 law. When working in the topological setup we have to modify the definition of Liouville operators by saying that an invariant Markov operator is *topologically fiberwise Liouville* if *all* (rather than almost all as in Definition 4.1) fiberwise operators $P_x : L^\infty(\mathbf{G}^x, \lambda^x) \hookleftarrow$ are Liouville.

Theorem 6.1. *Let* $\mathbf{G}$ *be a locally compact topological groupoid with a continuous Haar system, and let* P *be an invariant Markov operator on* $\mathbf{G}$ *with continuous densities. If the operator* P *is topologically fiberwise Liouville, then the groupoid* $\mathbf{G}$ *is topologically amenable.*

Proof. We shall use the fact that the Cesaro averages in the formulation of the 0–2 law for the triviality of the Poisson boundary can be replaced with any sequence of probability measures on $\mathbb{Z}_+$ strongly convergent to invariant mean on $\mathbb{Z}$ [Kai92]. By taking for such a sequence the binomial distributions on $\mathbb{Z}_+$ the convergence in formula (4.3) can be made monotone and therefore uniform on compacts.

More precisely, let us consider the Markov operator $Q = (P + P^2)/2$. The transition probabilities of the operator Q are the averages of the time 1 and time 2 transition probabilities of the operator P. Obviously, the operator Q is also invariant, and it is fiberwise Liouville simultaneously with the operator P. For the operator Q the Poisson boundary coincides with the tail boundary, and therefore the corresponding 0–2 law (see [Kai92]) implies that

$$\left\| \left(\delta_{\mathbf{g}} - \delta_{\mathbf{t}(\mathbf{g})}\right) Q^n \right\| \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall\, \mathbf{g} \in \mathbf{G}\,, \tag{6.2}$$

or, in other words, that

$$\left\| 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \left(\delta_{\mathbf{g}} - \delta_{\mathbf{t}(\mathbf{g})}\right) P^{n+k} \right\| \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall\, \mathbf{g} \in \mathbf{G}\,.$$

It is clear that the convergence in formula (6.2) is monotone. On the other hand, continuity of densities of the transition probabilities of the operator P (therefore, of its powers as well) and continuity of the Haar system implies that the left-hand side of (6.2) depends on $\mathbf{g}$ continuously. It remains to refer to Dini's theorem, according to which monotone convergence of continuous functions to a continuous limit on a compact set is uniform, and to conclude in the same way as in the proof of Theorem 4.2. □

6.B. **Amenability of group actions.** In the same way as in Section 4, we shall now give a specialization of Theorem 6.1 to the particular case of (groupoids associated with) group actions. In this situation topological amenability of a continuous action of a locally compact group G on a locally compact space X is equivalent to existence of a sequence of weak* continuous maps θ_n from the action space X to the space $\mathcal{P}(G)$ of probability measures on G which are *topologically approximatively equivariant* in the sense that $\|g\theta_n(x) - \theta_n(gx)\| \to 0$ uniformly on compact subsets of $G \times X$ (cf. Section **2.C**).

Theorem 6.3. *Let G be a locally compact group acting continuously and properly on a locally compact space S, let m be a G-invariant Borel measure on S, and let P be a G-invariant Markov operator on S with absolutely continuous transition probabilities and continuous densities with respect to the measure m. Suppose that the group G also acts continuously on another locally compact space X, and there is a continuous G-equivariant map assigning to points $x \in X$ projective classes of minimal positive P-harmonic functions φ_x. Then the action of G on X is topologically amenable.*

Proof. As in the proof of Theorem 6.1, let us pass from the operators $P_x = P_{\varphi_x}$ to the operators $Q_x = (P_x + P_x^2)/2$, and consider on S the probability measures

$$\theta_n(s, x) = \delta_s Q_x^n , \qquad s \in X,\ x \in X .$$

The maps $(s, x) \to \theta_n(s, x)$ are G-equivariant, and

$$\|\theta_n(s, x) - \theta_n(s', x)\| \underset{n\to\infty}{\longrightarrow} 0$$

uniformly on compact sets. Then for any fixed point $o \in S$

$$\|g\theta_n(o, x) - \theta_n(o, gx)\| = \|\theta_n(go, gx) - \theta_n(o, gx)\| \underset{n\to\infty}{\longrightarrow} 0$$

for any $g \in G$ and $x \in X$ uniformly on compact sets. Since the action of G on S is proper, the measures $\theta_n(o, x)$ can be equivariantly and continuously lifted from S to G [ADR00, Corollary 2.1.17], to provide a topologically approximatively equivariant sequence of maps from X to $\mathcal{P}(G)$. □

Remark 6.4. In [BG02] the following particular cases of Theorem 6.3 were established:

(i) G is a finitely generated group, $S = G$, and P is the Markov operator of a finitely supported random walk on G;

(ii) G is a lattice in an ambient locally compact group S, and P is the Markov operator of an absolutely continuous random walk on S.

Although the proof in [BG02] was also based on using the 0-2 law, it was done in the measure theoretical setting only by applying then the theorem of Anantharaman-Delaroche and Renault on the equivalence of the topological amenability and the *measurewise amenability* (i.e., the amenability of the measured groupoid $(\mathbf{G}, \lambda, \mu)$ for any measure μ on $\mathbf{G}^{(0)}$ quasi-invariant with respect to a fixed Haar system λ) for locally compact groupoids with a continuous Haar system and countable orbits [ADR00, Theorem 3.3.7].

6.C. Spaces of minimal harmonic functions. As we have already mentioned in Section **5.A**, the Radon–Nikodym derivatives of harmonic measures on the Poisson boundary of a Markov operator with absolutely continuous transition probabilities are minimal harmonic functions. The *Martin boundary* is a topological counterpart of the Poisson boundary. Unlike the Poisson boundary (defined as a measure space), the Martin boundary is a *bona fide* topological space obtained by taking the closure of the topological state space embedded into the space of functions on itself via the Green kernel under suitable regularity conditions on transition probabilities [Rev84]. The Martin boundary contains all minimal harmonic functions (the closure of the corresponding subset is sometimes called the *minimal Martin boundary*). Since the space of positive harmonic functions is a lattice, any harmonic function can be uniquely decomposed as an integral of minimal ones. Note that the Martin boundary considered as a measure space endowed with the representing measures of the constant function **1** coincides with the Poisson boundary. The action of any symmetry group of the Markov operator extends to the Martin boundary. See [Kai96] and the references therein for a discussion of the Martin boundary for Markov operators on homogeneous spaces.

Theorem 6.3 immediately implies:

Theorem 6.5. *Under conditions of Theorem 6.3, if the subset M of minimal harmonic functions in the Martin boundary of the operator P is closed, then the action of G on M is topologically amenable.*

Identification of the space of minimal harmonic functions of a G-invariant Markov operator is, in general, a difficult problem (e.g., see [Kai96]). Note, however, that there is no need for the space X from Theorem 6.3 to represent *all* minimal harmonic functions. Appropriate geometrical boundaries were shown to produce in a continuous way minimal harmonic functions (not necessarily all of them!) in several situations of hyperbolic flavour:

(i) If X is a *Gromov hyperbolic Riemannian manifold* of bounded geometry with a spectral gap, P is the Markov operator corresponding to the *Brownian motion* on X, and ∂X is the *hyperbolic boundary* of X [Anc90], in particular, if X is a *simply connected Riemannian manifold with pinched sectional curvatures* and ∂X is its *visibility boundary* [AS85];

(ii) If X is a *Gromov hyperbolic graph* satisfying the strong isoperimetric inequality, P is the Markov operator of the *simple random walk* on X, and ∂X is the *hyperbolic boundary* of X [Anc90];

(iii) If X is a *non-compact Riemannian symmetric space*, P is the Markov operator corresponding to the *Brownian motion* on X, and ∂X is the *Furstenberg boundary* of X (it is defined as the space of *asymptotic classes of Weyl chambers* in X, or, equivalently, as the quotient of the corresponding semi-simple Lie group G by a minimal parabolic subgroup; for $G = SL(n, \mathbb{R})$ this is the *flag space* in $\mathbb{R}^n$) [Fur63], [Kar67], see the book [GJT98] for the latest developments, in particular, for a description of minimal harmonic functions for random walks on general unimodular groups with Gelfand pairs (Theorem 13.12);

(iv) If X is a locally finite *affine building*, P is the Markov operator of the *simple random walk* on its set of vertices, and ∂X is the *spherical building at infinity* of X (the space of *asymptotic classes of sectors*); this case requires more explanations, so that its discussion is relegated to Section **6.D** below.

In all these cases the operator P agrees with the underlying geometrical structure on X, so that it commutes with the (proper) group of isomorphisms of X. Theorem 6.3

then provides topological amenability of the corresponding boundary actions "for free", which gives a unified generalization of numerous earlier results on amenability of boundary actions [Bow77], [Ver78], [Zim84], [Spa87], [SZ91], [Ada94], [Ada96], [RS96], [RR96], [CR03]:

Theorem 6.6. *The action of a closed group of isomorphisms of the space X on the boundary ∂X is topologically amenable in the above cases (i) – (iv).*

6.D. **Harmonic functions on buildings.** The set V of vertices of a locally finite affine building X is split into several types, so that by taking this additional structure into account one can naturally define several commuting Markov operators ("Laplacians") $P_1, P_2, \dots, P_d$ on V (where d is the dimension of the building; see the references below for details). By the *simple random walk* on V we shall mean the Markov chain associated with the operator $P = (P_1 + P_2 + \cdots + P_d)/d$.

A function f on V is called *strongly harmonic* if it is harmonic in the usual sense ($P_i f = f$) for all operators P_i. It is known that there is a continuous map assigning to points $\gamma \in \partial X$ functions φ_γ on V which are strongly harmonic and minimal in the cone of non-negative strongly harmonic functions (minimality follows from the uniqueness of the boundary decomposition for strongly harmonic functions). More precisely, it follows from the results of Kato [Kat81] (also see [GJT98, Theorem 13.12]) that this is true for affine buildings associated with reductive groups over p-adic fields, in particular, for all buildings of dimension at least 3 [Tit86]. Dimension 1 affine buildings are just trees, whereas for the remaining "non-classical" affine buildings of dimension 2 it was proved by Mantero and Zappa [MZ98], [MZ00], [MZ02] (also see [Car99] for a unified treatment of type $\widetilde{A}_n$ buildings).

Obviously, any strongly harmonic function is also P-harmonic. Although the converse is not true in general, one can show that *any function which is minimal in the cone of non-negative strongly harmonic functions is also minimal in the cone of non-negative P-harmonic functions* (which, in particular, implies coincidence of P-harmonicity and strong harmonicity for bounded functions, see Remark 6.7 below). Thus, the above functions φ_γ are minimal P-harmonic, which is precisely what is needed for applying Theorem 6.3.

We shall briefly outline the proof (which actually works for any Markov operator P from the "Hecke algebra" generated by a family of commuting Markov operators $\{P_i\}$ under suitable non-degeneracy conditions), the details to be given elsewhere. Let φ be a minimal strongly harmonic function. By passing from the operators P_i and P to their Doob transforms (which commute simultaneously with the original operators) we may assume that $\varphi = \mathbf{1}$. If $\mathbf{1}$ is not minimal P-harmonic, then the Poisson boundary of the operator P is non-trivial. Let ψ be the bounded P-harmonic function which corresponds to a non-constant measurable function $\widehat{\psi}$ on the Poisson boundary with the values 0 and 1 only. By the martingale convergence theorem the values of ψ converge to the function $\widehat{\psi}$ along a.e. sample path (x_n) of the Markov chain associated with the operator P, so that these limits are either 0 or 1. On the other hand,

$$\psi(x_n) = P\psi(x_n) = \frac{1}{d}\big(P_1\psi(x_n) + \cdots + P_d\psi(x_n)\big) .$$

Since the operators P_i and P commute, the functions $P_i\psi$ are also P-harmonic and take values between 0 and 1. Therefore, their limits along almost all sample paths are the same as for the function ψ. Since bounded harmonic functions are determined by their limit values on the Poisson boundary, we conclude that $P_i\psi = \psi$, i.e., ψ is strongly harmonic

in contradiction to the hypothesis of minimality of the constant function **1** in the cone of strongly harmonic functions.

Remark 6.7. The functions φ_γ, $\gamma \in \partial X$ provide a decomposition of the constant function. Since, as we have just proved, φ_γ are minimal P-harmonic, *the space ∂X with the corresponding representing measure is the Poisson boundary of the operator P*. In particular, *for bounded functions on X strong harmonicity is equivalent to P-harmonicity* (cf. [MZ03] for the dimension 2 case).

6.E. Amenability at infinity. A locally compact group G is called *amenable at infinity* if it admits a topologically amenable action on a Hausdorff compact space X, i.e., if the associated groupoid $G \ltimes X$ is topologically amenable. For a countable group G its amenability at infinity is equivalent to topological amenability of its action on the Stone-Čech compactification βG, and, moreover, if G is finitely generated, to existence of a uniform embedding of its Cayley graph into Hilbert space [HR00]. This notion has found important applications in the theory of C^* algebras, see [HR00], [Hig00], [AD02], [Val02], [CEOO03] and the references therein. Theorem 6.6 implies

Theorem 6.8. *Any closed subgroup of the group of isometries of any of the spaces listed in Theorem 6.6 is amenable at infinity.*

It is known that any discrete subgroup of a connected Lie group is amenable at infinity [ADR00, Example 5.2.2]. As it follows from the theorem of Adams on universal amenability of the boundary action of the group of isometries of an exponentially bounded Gromov hyperbolic space [Ada96], any discrete group of isometries of such a space is also amenable at infinity in view of [ADR00, Theorem 3.3.7] (also see [Ger00] for the particular case of word hyperbolic groups). It was proved in [Kai03] that this is true for general closed groups of isometries of Gromov hyperbolic spaces as well under suitable bounded geometry assumptions (without which amenability of the boundary action may fail).

Remark 6.9. Disproving a conjecture from [HR00], Gromov [Gro00] showed that there exist finitely generated groups G whose Cayley graph does not admit a uniform embedding into Hilbert space, and which, therefore, are not amenable at infinity. In view of Theorem 6.5 these groups have a curious property: the set of minimal harmonic functions in the Martin boundary of any random walk on G (more generally, on any proper G-space) is never closed.

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